stichting
mathematisch
centrum

AFDELING MATHEMATISCHE BESLISKUNDE (DEPARTMENT OF OPERATIONS RESEARCH) BW 144/81 JUNI

M.H. VAN HOORN & H.C. TIJMS

APPROXIMATIONS FOR THE WAITING TIME DISTRIBUTION OF THE M/G/c QUEUE

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 60K25

Approximations for the waiting time distribution of the M/G/c queue*)

by

M.H. van Hoorn**) & H.C. Tijms***)

ABSTRACT

For the M/G/c queue we present an approximate analysis of the waiting time distribution. The result is given in the form of a defective renewal equation. This integral equation can be numerically solved by a simple recursion procedure. Also, asymptotic results for the waiting times are presented. Numerical results indicate that the approximations are sufficiently accurate for practical purposes.

KEY WORDS & PHRASES: *M/G/c queue; waiting time distribution; approximations.*

---

*) This paper will be submitted for publication elsewhere.

**) Vrije Universiteit, Amsterdam.

***) Advisor Mathematical Centre. Address: Vrije Universiteit, Amsterdam.

1. INTRODUCTION

In recent years considerable attention has been paid to the development of approximations for various operating characteristics of the M/G/c queue. In particular several good approximations for the mean queue size have been developed, see BOXMA, COHEN and HUFFELS (1980), COSMETATOS (1976), TAKAHASHI (1977), TIJMS, VAN HOORN and FEDERGRUEN (1981) amongst others. Approximations for the state probabilities have been given in HOKSTAD (1978) and TIJMS et al (1981). The various approximations suggested in TIJMS et al (1981) are computed by a stable recursive algorithm and include the approximation of HOKSTAD (1978) as a special case. It turned out from extensive numerical comparisons that one of the new approximations in TIJMS et al (1981) was in general superior to the other ones. Using this particular approximation for the queue size distribution, we shall develop an accurate approximation for the waiting time distribution under the assumption that service is in order of arrival. The approximation for the waiting-time distribution will be given in the form of a defective renewal equation. This integral equation is very well suited to be solved after discretisation by a stable forward recursive algorithm for all values of the queueing parameters.

For deterministic service times an exact method has been given in CROMMELIN (1932), but this method is of practical use only for smaller values of the number of servers, cf. also KÜHN (1976). For phase type service time an asymptotic formula for the tail of the waiting time distribution was recently obtained in TAKAHASHI (1980), cf. also NEUTS and TAKAHASHI (1980). Also, this expression is only to a limited extent useful for practical purposes since the coefficients of the asymptotic formula require the solving of the balance equations for the state probabilities in the multidimensional Markov chain representation of the queueing process. Finally, for special cases of the M/G/c queue exact methods for the waiting time distribution have been discussed in AVIS (1976) and COHEN (1980), but these methods are not very suitable for practical purposes.

In section 2 we shall derive the integral equation to compute the approximate waiting times and in section 3 we present some numerical results.

2. THE INTEGRAL EQUATION FOR THE APPROXIMATE WAITING TIME DISTRIBUTION.

Consider the M/G/c queue with $c \geq 1$ servers and an infinite waiting capacity. Customers arrive according to a Poisson process with rate λ and the service time S of a customer has a general probability distribution function $F(t) = \Pr\{S \leq t\}$ with $F(0) = 0$. It is assumed that the traffic intensity $\rho = \lambda ES/c$ is less than 1.

Define the following quantities, assuming that the system is in the steady state.

p_j = probability that at an arbitrary epoch j customers are in the system,

L_q = the number of customers in the queue at an arbitrary epoch (excluding any customer in service),

W_q = the amount of time spent in the queue by an arbitrary customer (excluding his service time),

$W_q(t) = \Pr\{W_q \leq t\}$.

Observe that by the assumption of Poisson arrivals

p_j = probability that an arbitrary customer sees upon arrival j other customers in the system

i.e. Poisson arrivals see time averages (cf. STIDHAM(1972)). Hence the delay probability P_W is given by

$$P_W = \Pr\{W_q > 0\} = \sum_{j=c}^{\infty} p_j.$$

Further, define the equilibrium distribution F_e of F by

$$F_e(t) = \frac{1}{ES} \int_0^t (1-F(x))\,dx .$$

In TIJMS et al (1981) the following recursive scheme was derived to compute approximations $\bar{p}_j$, $j \geq 0$ for the state probabilities p_j (cf. also TIJMS and van HOORN (1981))

$$\bar{p}_n = \frac{(\lambda ES)^n}{n!} \bar{p}_0 \quad , \quad 0 \leq n \leq c-1,$$

$$\bar{p}_n = \lambda\, \bar{p}_{c-1}\, \alpha_{n-c} + \lambda \sum_{j=c}^{n} \bar{p}_j\, \beta_{n-j} \quad , \ n \geq c,$$

where

$$\bar{p}_0 = 1/\left(\sum_{j=0}^{c-1} \frac{(\lambda ES)^j}{j!} + \frac{(\lambda ES)^c}{c!(1-\rho)}\right),$$

$$\alpha_n = \int_0^\infty (1-F_e(t))^{c-1}(1-F(t))\, e^{-\lambda t} \frac{(\lambda t)^n}{n!}\, dt \quad , \; n \geq 0,$$

$$\beta_n = \int_0^\infty (1-F(ct))\, e^{-\lambda t} \frac{(\lambda t)^n}{n!}\, dt \quad , \; n \geq 0.$$

In general the numbers α_n and β_n have to be computed by numerical integration, but for several special cases of the service time distribution explicit expressions can be given. Note that the corresponding approximation for the delay probability P_W is given by

$$\bar{P}_W = \frac{(\lambda ES)^c}{c!(1-\rho)}\, \bar{p}_0 = \frac{\rho}{1-\rho}\, \bar{p}_{c-1} \; ,$$

i.e. the widely used Erlang delay probability approximation. Further, using generating functions and the relation $EL_q = \lambda\, EW_q$ we obtain for EW_q the approximation

$$\overline{EW}_q = \{\rho \frac{ES^2}{2(ES)^2} + (1-\rho)\frac{c}{ES} \int_0^\infty (1-F_e(t))^c\, dt\}\; EW_q(\exp).$$

This approximation is exact for both $\rho \to 0$ and $\rho \to 1$, cf. BOXMA et al (1980), KÖLLERSTROM (1974) and BURMAN and SMITH (1981).

We now turn to the determination of an approximation for the waiting time distribution. Therefore we assume that customers are served *in order of arrival*. Consider a test customer. The test customer sees upon arrival L_{q_1} customers in the queue, has waiting time W_q and leaves upon entering service L_{q_2} customer behind in the queue. By an up-and down crossing argument observe that L_{q_1} and L_{q_2} have the same distribution, whereas the L_{q_2} customers have been arrived during W_q. By the assumption of Poisson arrivals L_{q_1} has the same distribution as L_q. Hence the number of customers arrived during W_q has the same distribution as L_q and so we have the known relation (cf. MARSHALL and WOLFF (1971) and HAJI and NEWELL (1971)),

$$(1) \qquad E z^{L_q} = \sum_{j=0}^{\infty} z^j \Pr\{L_q = j\} = \sum_{j=0}^{\infty} z^j \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} d\Pr\{W_q \le t\} =$$

$$= E\, e^{-\lambda(1-z)W_q} .$$

Noting that $\Pr\{L_q = 0\} = \sum_{j=0}^{c} p_j = 1 - P_W + p_c$ and $\Pr\{L_q = j\} = P_{c+j}, j \ge 1$ and using the approximations for p_j it is straight forward to show

$$(2) \qquad \overline{E z^{L_q}} = 1 - \bar{P}_W + \lambda \bar{p}_{c-1} \frac{\int_0^{\infty} (1-F_e(t))^{c-1}(1-F(t))\, e^{-\lambda(1-z)t}\, dt}{1 - \int_0^{\infty} (1-F(ct))\, e^{-\lambda(1-z)t}\, dt} .$$

For clarity of presentation we define the probability distribution functions

$$G(t) = 1 - (1 - F_e(t))^c , \quad H(t) = \frac{1}{ES} \int_0^{ct} (1-F(x))\, dx, \quad t \ge 0$$

By combining (1) and (2), we find for the Laplace transform of $\bar{W}_q$

$$(3) \qquad E\, e^{-s\bar{W}_q} = 1 - \bar{P}_W + \rho \bar{p}_{c-1} \frac{\int_0^{\infty} e^{-st}\, dG(t)}{1 - \rho \int_0^{\infty} e^{-st}\, dH(t)} .$$

Inversion of (3) gives the Pollaczek-Khintchine like result

$$\bar{W}_q(t) = 1 - \bar{P}_W + \rho \bar{p}_{c-1} \sum_{n=0}^{\infty} \rho^n (G * H^{n*})(t), \quad t \ge 0,$$

where $*$ denotes the convolution operator and $H^{0*}(t) \equiv 1$. Define

$$V(t) = 1 - P\{W_q > t \mid W_q > 0\}, \quad t \ge 0,$$

i.e. $V(t)$ is the waiting time distribution for the delayed customers. Noting that $V(t) = 1 - (1 - W_q(t))/P_W$ and using $\rho \bar{p}_{c-1} = (1-\rho)\, \bar{P}_W$, we get the approximate result

$$(4)\qquad \bar{V}(t) = (1-\rho)\sum_{n=0}^{\infty} \rho^n (G * H^{n*})(t),\ t\geq 0.$$

By taking the convolution of (4) with H we get the defective renewal equation

$$\bar{V}(t) = (1-\rho)G(t) + \rho\int_0^t \bar{V}(t-x)\,dH(x),\ t\geq 0,\ \text{or}$$

$$(5)\qquad \bar{V}(t) = (1-\rho)\{1-(1-F_e(t))^c\} + \lambda\int_0^t \bar{V}(x)(1-F(c(t-x))\,dx,\ t\geq 0.$$

In the appendix we discuss a numerical procedure to solve (5).

REMARK. *Asymptotic results for the waiting times*

Following the analysis on p. 362 in FELLER (1966), we can reduce the defective renewal equation (5) to a proper renewal equation and apply the key renewal theorem to this latter equation. Thus we obtain

$$(6)\qquad \bar{V}(t) \sim \frac{(1-\rho)\int_0^\infty e^{\kappa y}\{1-(1-F_e(y))^c\}dy}{\lambda\int_0^\infty y\,e^{\kappa y}(1-F(cy))dy}\, e^{-\kappa\tau}\quad,\ t\to\infty$$

where $\kappa > 0$ is the unique solution to

$$\lambda\int_0^\infty e^{\kappa y}(1-F(cy))dy = 1.$$

This approximate asymptotic expansion is very close to the exact asymptotic expansion of $P\{W_q>t\}$ derived in TAKAHASHI (1980). For the case of a phase-type service time distribution it was shown in TAKAHASHI (cf. also NEUTS and TAKAHASHI (1980)),

$$(7)\qquad P\{W_q > t\} \sim \frac{\xi\pi_0}{\lambda(\tau-1)^2\tau^{c-1}}\, e^{-\xi t}\quad,\ t\to\infty,$$

where $\xi > 0$ is the unique solution to $\int_0^\infty e^{\xi y/c}\,dF(y) = 1+\xi/\lambda$, $\tau = 1+\xi/\lambda$ with $\lim_{n\to\infty} p_n/p_{n-1} = \tau$ and $\pi_0 = \lim_{n\to\infty} \tau^n p_n$. It is easily shown that

$\kappa = \xi$ and hence the asymptotic formulas (6) and (7) are identical except for a multiplicative constant. We finally remark that using the discrete renewal theorem it can be readily verified from our recursion relation for the $\bar{p}_n$ that $\lim_{n\to\infty} \bar{p}_n/\bar{p}_{n-1} = \tau$ and hence is exact.

3. NUMERICAL RESULTS.

In this section we present some numerical results for the waiting time distribution. We have made the following choices for the distribution of the service time S.

1) deterministic (D),
2) Erlang-2 (E_2), density $\mu^2 te^{-\mu t}$, $\mu = 2$, $cv^2 = 0.5$,
3) mixture of Erlang-1 and Erlang-3 ($E_{1,3}$), density $p\mu e^{-\mu t}+(1-p)\frac{1}{2}\mu^3 t^2 e^{-\mu t}$, $p = 0.225708$, $\mu = 3-2p$, $cv^2 = 0.5$,
4) hyperexponential (H_2), density $p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}$, $p = 0.810087$, $\mu_1 = 2p$, $\mu_2 = 2(1-p)$, $cv^2 = 2.25$.

The mean service time is taken to be 1 and cv^2 denotes the squared coefficient of variation of S, i.e. $cv^2 = ES^2/(ES)^2 - 1$.
We have solved the integral equation for the approximate waiting time distribution by using the numerical procedure given in the appendix. In case 1) we have compared the approximate results (app.) with the exact results (ex) of Kühn (1976). In the other cases we compare our approximate results with the asymptotic results (asy) of Takahashi (1980) and with simulation results (sim). The difficulty in computing the asymptotic results is the determination of the constant π_0 in (7). Therefore the exact values of the state probabilities p_n have to be computed and this is only computationally feasible for smaller values of c. For our numerical examples we have used the decomposition method of Takahashi and Takami (1976) to compute the exact values of the state probabilities. For each example we have simulated **one million** customers. In the tables the notation .77(1) means that the 95% confidence interval of the simulated value is .76-.78. The tables 1 and 2 indicate that the approximate results are accurate enough for practical purposes and are at least as accurate as the results obtained by time-consuming computer simulation. The computation time for the approximate results was about 1 second CPU time for each example and was practically independent of the values of c, ρ and cv^2. The asymptotic results required per example between 1 and 15 seconds CPU time whereas the simulation of one example with one million customers took on the average 180 seconds CPU time.

Table 1. $P\{W_q > T \mid W_q > 0\}$, $\rho=0.8$

T	0.1	0.25	0.5	0.75	1.0	1.5	2.0	3.0	
c= 3									
app.	.9385	.8371	.6422	.4696	.3399	.1781	.0933	.0256	D
ex.	.9277	.8123	.6146	.4396	.3172	.1666	.0874	.0240	
app.	.9390	.8461	.6996	.5729	.4675	.3105	.2061	.0907	
asy.	.9697	.8574	.6985	.5690	.4636	.3076	.2042	.0899	E_2
sim.	.94(1)	.85(1)	.70(1)	.57(1)	.47(1)	.31(1)	.21(1)	.095(6)	
app.	.9403	.8511	.7076	.5801	.4730	.3132	.2072	.0907	
asy.	.9832	.8685	.7064	.5746	.4673	.3092	.2045	.0895	$E_{1,3}$
sim.	.93(1)	.84(1)	.69(1)	.56(1)	.46(1)	.30(1)	.20(1)	.083(5)	
app.	.9429	.8677	.7656	.6849	.6188	.5139	.4311	.3061	
asy.	.8204	.7796	.7162	.6578	.6043	.5099	.4302	.3063	H_2
sim.	.95(1)	.88(1)	.79(1)	.71(1)	.64(1)	.53(1)	.44(1)	.31(1)	
c= 5									
app.	.8979	.7190	.4428	.2597	.1516	.0516	.0176		D
ex.	.8769	.6917	.4204	.2409	.1413	.0481	.0164		
app.	.8996	.7521	.5422	.3864	.2747	.1387	.0701	.0179	
asy.	.9347	.7614	.5410	.3844	.2732	.1379	.0696	.0178	E_2
sim.	.89(1)	.75(1)	.54(1)	.38(1)	.27(1)	.137(6)	.069(5)	-	
app.	.9019	.7589	.5495	.3912	.2775	.1394	.0700	.0177	
asy.	.9514	.7738	.5484	.3887	.2755	.1384	.0695	.0175	$E_{1,3}$
sim.	.90(1)	.75(1)	.54(1)	.39(1)	.27(1)	.137(7)	.068(5)	-	
app.	.9071	.7935	.6554	.5558	.4775	.3575	.2691	.1527	
asy.	.7597	.6979	.6058	.5258	.4564	.3439	.2591	.1471	H_2
sim.	.91(1)	.80(1)	.67(1)	.56(1)	.48(1)	.35(1)	.26(1)	.14(1)	
c=10									
app.	.7881	.4590	.1606	.0547	.0186	.0022			D
ex.	.7599	.4553	.1578	.0533	.0180	.0021			
app.	.8041	.5493	.2793	.1411	.0712	.0182	.0046		
asy.	.8459	.5614	.2835	.1431	.0723	.0184	.0047		E_2
sim.	.81(1)	.56(1)	.29(1)	.149(7)	.076(6)	.020(4)	-		
app.	.8084	.5557	.2821	.1418	.0712	.0180	.0045		
asy.	.8683	.5744	.2885	.1449	.0728	.0184	.0046		$E_{1,3}$
sim.	.81(1)	.56(1)	.30(1)	.15(1)	.077(7)	.019(4)	-		
app.	.8263	.6510	.4732	.3542	.2666	.1513	.0859	.0277	
asy.	.6392	.5393	.4064	.3062	.2307	.1309	.0743	.0239	H_2
sim.	.85(1)	.68(1)	.47(1)	.34(1)	.25(1)	.137(8)	.076(7)	.023(5)	

Table 2. $P\{W_q > T \mid W_q > 0\}$,c=5

T	0.1	0.25	0.5	0.75	1.0	1.5	2.0	3.0	
ρ=0.5									
app.	.7653	.4475	.1311	.0293	.0061				D
ex.	.7366	.4231	.1192	.0215	.0046				
app.	.7682	.4927	.2160	.0902	.0370	.0062			
asy.	.9668	.5638	.2294	.0934	.0380	.0063			E_2
sim.	.77(1)	.49(1)	.216(7)	.092(5)	.038(4)	-			
app.	.7730	.5043	.2240	.0928	.0374	.0060			
asy.	1.060	.6096	.2425	.0965	.0384	.0061			$E_{1,3}$
sim.	.76(1)	.49(1)	.22(1)	.096(5)	.040(3)	-			
app.	.7834	.5586	.3450	.2308	.1619	.0847	.0454		
asy.	.3351	.2788	.2052	.1511	.1112	.0602	.0326		H_2
sim.	.80(1)	.58(2)	.36(1)	.23(1)	.15(1)	.073(7)	.037(5)		
ρ=0.7									
app.	.8511	.6115	.2932	.1277	.0549	.0101	.0019		D
ex.	.8244	.5809	.2722	.1122	.0489	.0090	.0017		
app.	.8534	.6527	.3988	.2387	.1421	.0502	.0177	.0022	
asy.	.9213	.6743	.4008	.2383	.1416	.0500	.0177	.0022	E_2
sim.	.85(1)	.65(1)	.40(1)	.24(1)	.144(7)	.052(4)	.019(2)	-	
app.	.8566	.6616	.4069	.2429	.1438	.0502	.0175	.0021	
asy.	.9526	.6946	.4102	.2423	.1431	.0499	.0174	.0021	$E_{1,3}$
sim.	.85(1)	.65(1)	.40(1)	.242(5)	.143(4)	.050(3)	.017(2)	-	
app.	.8639	.7063	.5302	.4153	.3327	.2190	.1456	.0646	
asy.	.6164	.5457	.4456	.3638	.2970	.1979	.1319	.0586	H_2
sim.	.87(1)	.73(1)	.55(1)	.42(1)	.33(1)	.21(1)	.14(1)	.059(7)	
ρ=0.9									
app.	.9475	.8474	.6673	.5159	.3982	.2372	.1413	.0063	D
ex.	.9354	.8297	.6498	.4985	.3854	.2297	.1369	.0061	
app.	.9484	.8671	.7368	.6231	.5265	.3758	.2682	.1366	
asy.	.9621	.8695	.7346	.6206	.5243	.3742	.2671	.1360	E_2
sim.	.94(1)	.86(1)	.73(1)	.62(1)	.52(2)	.37(2)	.26(1)	.13(1)	
app.	.9496	.8709	.7417	.6272	.5297	.3776	.2692	.1368	
asy.	.9690	.8754	.7391	.6240	.5269	.3756	.2678	.1361	$E_{1,3}$
sim.	.95(1)	.87(1)	.74(1)	.63(1)	.53(1)	.38(1)	.27(1)	.14(1)	
app.	.9525	.8909	.8096	.7446	.6888	.5927	.5111	.3803	
asy.	.8893	.8507	.7901	.7339	.6816	.5879	.5071	.3774	H_2
sim.	.95(1)	.89(1)	.81(1)	.74(1)	.68(2)	.58(2)	.49(2)	.36(2)	

APPENDIX

For the numerical solution of the integral equation (5) we propose the following procedure, which can be found in detail in DELVES and WALSH (1974). Consider the integral equation

$$f(t) = g(t) + \int_0^t f(x)\,k(t-x)dx \ , \ t \geq 0$$

to be solved for f(t), where g(t) and k(t) are known differentiable functions. Choose a step length h and let f_n denote f(nh), etc. Beside $f_0(= g_0)$ a second initial value f_1 can be obtained using Day's starting procedure. Letting $g_{\frac{1}{2}}=g(h/2)$ and $k_{\frac{1}{2}}=k(h/2)$, define

$$a_1 = g_1 + hg_0 k_1$$

$$a_2 = g_1 + \tfrac{1}{2} h (g_0 k_1 + a_1 k_0)$$

$$a_3 = g_{\frac{1}{2}} + \tfrac{1}{4} h(g_0 k_{\frac{1}{2}} + \tfrac{1}{2} g_0 k_0 + \tfrac{1}{2} a_2 k_0)$$

then

$$f_1 = g_1 + \frac{1}{6} h(g_0 k_1 + 4a_3 k_{\frac{1}{2}} + a_2 k_0)$$

For the integration we use repeated Simpson's rule. We need to distinguish between n even and n odd.

n even: $$f_n = g_n + \frac{1}{3} h \sum_{j=0}^{n} d_{n,j}\, f_j\, k_{n-j} \ ,$$

n odd : $$f_n = g_n + \frac{1}{3} h \sum_{j=0}^{n-3} d_{n-3,j}\, f_j\, k_{n-j} + \frac{3}{8} h(f_{n-3} k_3 + 3 f_{n-2} k_2 + 3 f_{n-1} k_1 + f_n k_0),$$

where $d_{n,j} = 3-(-1)^j$, $1 \leq j \leq n-1$, $d_{n,0} = d_{n,n} = 1$ are the weights of the integration rule. We remark that for the M/D/c case only slight modifications in the above procedure are required since the functions g(t) and h(t) are then only piece-wise differentiable.

ACKNOWLEDGEMENTS

We would like to thank Harrie Groenevelt for his help in writing the computer programs and Herman te Riele for helpful discussions about solving integral equations.

REFERENCES

[1] AVIS, D. (1977), *Computing waiting times in* $GI/E_k/c$ *queueing systems*, TIMS Studies in Management Sciences 7, 215-232.

[2] BOXMA, O.J., COHEN, J.W. & N. HUFFELS (1980), *Approximations of the mean waiting time in an* M/G/s *queueing system*, Op. Res. 27, 1115-1127.

[3] BURMAN, D.Y. & D.R. SMITH (1981), *A light traffic theorem for multi server queues*, Report Bell Labs, Holmdel, New Jersey.

[4] COHEN, J.W. (1979), *On the* M/G/2 *queueing model*, (to appear in Stoch. Proc. and Appl).

[5] COSMETATOS, G.P. (1976), *Some approximate equilibrium results for the multiserver queue* M/G/r, Op. Res. Quart. 27, 615-620.

[6] CROMMELIN, C.D. (1932), *Delay probability formulae when the holding times are constant*, P.O. Elect. Engrs J., 25, 41-50.

[7] DELVES, L.M. & J. WALSH (1974), *Numerical solution of integral equations*, Clarendon Press, Oxford.

[8] FELLER, W.F. (1966), *Introduction to probability theory and its applications*, vol. II, Wiley, New York.

[9] HAJI, R. & G.F. NEWELL (1971), *A relation between stationary queue and waiting time distributions*, J. Appl. Prob. 8, 617-620.

[10] HOKSTAD, P. (1978), *Approximations for the* M/G/m *queue*, Op. Res. 26, 511-523.

[11] KÖLLERSTROM, J. (1974), *Heavy traffic theory for queues with several servers*, J. Appl. Prob. 11, 544-552.

[12] KÜHN, P. (1976), *Tables on Delay Systems*, Inst. of Switching and Data Technics. University of Stuttgart.

[13] MARSHALL, K.T. & R.W. WOLFF (1971), *Customer average and time average queue lengths and waiting times*, J. Appl. Prob. 8, 535-542.

[14] NEUTS, M.F. & Y. TAKAHASHI (1980), *Asymptotic behavior of the stationary distributions in the* GI/Ph/c *queue with heterogeneous servers*, (to appear in Zeitschrift für Wahrscheinlichkeitsth).

[15] STIDHAM, S., Jr (1972), *Regenerative processes in the theory of queues with applications to the alternating priority queue*, Adv. Appl. Prob. 4, 542-557.

[16] TAKAHASHI, Y. & Y. TAKAMI (1976), *A numerical method for the steady-state probabilities of a* GI/G/c *queueing system in a general class*, J. Oper. Res. Soc. Japan, 19, 147-157.

[17] TAKAHASHI, Y. (1977), *An approximation formula for the mean waiting time of an* M/G/c *queue*, J. Op. Res. Soc. Japan 20, 150-163.

[18] TAKAHASHI, Y. (1980), *Asymptotic exponentiality of the tail of the waiting time distribution in a* Ph/Ph/c *queue*, Discussion paper 31, Dept. of Economics, Tokohu University, Japan.

[19] TIJMS, H.C.,& M.H. VAN HOORN (1981), *Computational methods for single servers and multi server queues with Markovian input and general service times*, Report BW 135, Mathematical Centre, Amsterdam.

[20] TIJMS, H.C., VAN HOORN, M.H. & A. FEDERGRUEN (1981), *Approximations for the steady state probabilities in the* M/G/c *queue*, Adv. Appl. Prob. 13, 186-206.